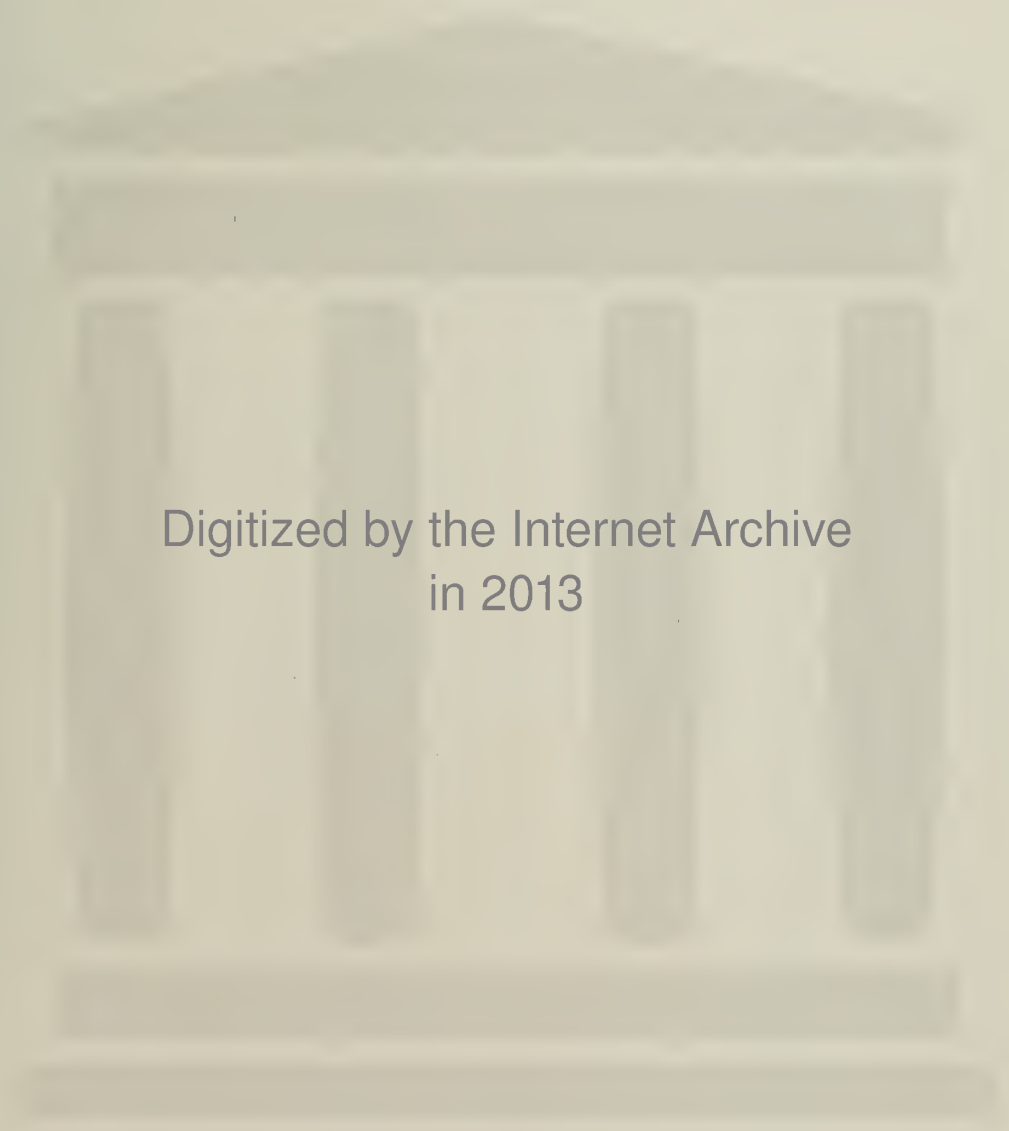

 Report No. 478

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by

MARTIN DIAMOND

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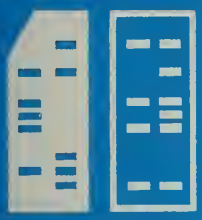

DEPARTMENT OF COMPUTER SCIENCE
UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN · URBANA, ILLINOIS

Report No. 478

THE DIVERGENCE OF STONE'S FACTORIZATIONS WHEN NO PARAMETERS ARE USED*

by

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September 7, 1971

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1. INTRODUCTION

The solution of a system of equations AX = q by factorization techniques can be separated into two segments. The first step is the definition of an auxiliary matrix, B, which is chosen so that A + B = LU where L and U are sparse lower and upper triangular matrices respectively. In practical computer programs the elements of B are not computed; instead the elements of L and U are computed. Then by using the factors L and U, systems of equations whose matrix of coefficients is A+B can be easily solved.

The second part of the method consists of an algorithm which uses an iteration of the form

$$(A+B)X_{n+1} = (A+B)X_n - (AX_n-q), \quad n = 0, 1, 2, \ldots$$

to generate a sequence of vectors $\{X_n\}$ which converges to the solution AX = q. The convergence of the iteration is dependent upon the definition of the auxiliary matrix (through the definition of L and U) and a sequence of parameters.

One of two types of parameters is employed in the iteration. The first type multiplies the term (AX_n-q) in the iteration as follows,

$$(A+B)X_{n+1} = (A+B)X_n - \tau_n(AX_n-q). \tag{1.1}$$

an analysis of this parameter and an algorithm based on the iteration (1.1) can be found in Diamond [1].

The second set of parameters may be used to alter the auxiliary matrix for each n, producing the iteration

$$(A+B_n)X_{n+1} = (A+B_n)X_n - (AX_n-q). \tag{1.2}$$

Although there are algorithms which use a sequence of auxiliary matrices; there has been no rigorous mathematical basis for the definition of an auxiliary matrix, or a sequence of matrices, which will make (1.2) rapidly convergent.

Stone has developed in [3] an algorithm of this type which has proved to be rapidly convergent on a number of test problems. The fundamental idea of Stone's factorizations is to design the auxiliary matrix so that the terms in each component of $B_n X$ cancel. When $B_n X_n$ and $B_n X_{n+1}$ are small it is intuitively plausible that one iteration of (1.2) will yield a vector which is approximately equal to the solution of $AX = q$.

This cancellation is the basis of the factorization; however it doesn't explain the convergence of the iteration. Indeed, experimental results have shown that repeatedly using the auxiliary matrix $B_{\alpha=1}$ (see 3.2 and 3.3 for precise definition) which produces the greatest canceling yields a divergent iteration.

In this paper we explain the divergence of iteration (1.2) if the auxiliary matrix $B_{\alpha=1}$ is used on every iteration. We then present some experimental results that indicate the eigenvalue of the iteration matrix which has largest modulus grows without bound as the dimension of A increases.

2. ORIGIN OF THE PROBLEM

Consider the Dirchlet problem

$$- \frac{\partial}{\partial X_1} (a_1(X) \frac{\partial U}{\partial X_1}) - \frac{\partial}{\partial X_2} (a_2(X) \frac{\partial U}{\partial X_2}) = q(X), \quad X \in D,$$

$$u(X) = \psi(X), \text{ for } X \in \partial D \tag{2.1}$$

where $x = (x_1, x_2)$, D is the interior of a compact region with boundary ∂D and where the differential operation in (2.1) is elliptic, that is $a_1(x)$ and $a_2(x)$ are strictly positive on D. A common method of solving (2.1) is to approximate the differential operator with a difference operator. A system of linear equations results and the problem is then to find a solution to the system of equations.

The system of equations is derived as follows. Assume D lies in the first quadrant; cover D with two families of parallel lines,

$$x_1 = jh \qquad j = 1, \ldots, n_1$$

and

$$x_2 = kh \qquad k = 1, \ldots, n_2.$$

Denote the points of intersection (jh, kh) by (j,k). The points (j,k) are called the mesh points or gridpoints. Let ∂D_h be the polygon formed by joining those mesh points (i,j) for which one of $(j \pm 1, k)$ or $(j, k \pm 1)$ is not in D. Similarly let D_h be the open and connected set bounded by ∂D_h. At each gridpoint of D_h the differential operator is approximated by a difference operator and the equations are in one-to-one correspondence with the gridpoints.

For simplicity of notation we shall consider the region D to be the unit square buonded by the x_1 and x_2 axes and the lines $x_1 = 1$ and $x_2 = 1$. Letting $x = (x_{1,1}, x_{1,2}, \cdots x_{n,1}, x_{2,1}, \ldots, x_{n,n})^T$ where $n_1 = n_2 = n$, we obtain the matrix equation

$$AX = q \tag{2.2}$$

defined by $(AX)_{j,k} = q_{j,k}$, where the subscript corresponds to the gridpoint (j,k) and $q_{j,k}$ is derived from $q(jh,kh)$ and the boundary conditions.

The difference operator can be chosen so that the matrix A is diagonally dominant and positive definite with $a_{i,i} > 0$ and $a_{i,j} \leq 0$ for $i = j$. One method for deriving such an A is to approximate $\frac{\partial}{\partial x_i}(a_i(x)\frac{\partial}{\partial x_i}u$ by

$$h^{-2}\{a_i(x+\tfrac{h}{2}e_i)[u(x) - u(x+he_i)] + a_i(x-\tfrac{h}{2}e_i)[u(x) - u(x-he_i$$

where e_1 and e_2 are unit vectors along the x_1 and x_2 axes respectively. This defines A by

$$(AX)_{j,k} = B_{j,k}x_{j,k-1} + D_{j,k}x_{j-1,k} + E_{j,k}x_{j,k} + D_{j+1,k}x_{j+1,k}$$

$$+ B_{j,k+1}x_{j,k+1}, \quad j,k = 1, \ldots, n, \text{ where}$$

$$B_{j,k} = -a_2(jh,(k-\tfrac{1}{2})h),$$

$$D_{j,k} = -a_1((j-\tfrac{1}{2})h,kh), \tag{2.3}$$

and

$$E_{j,k} = a_2(jh,(k-\tfrac{1}{2})h) + a_2(jh,(k+\tfrac{1}{2})h) + a_1((j-\tfrac{1}{2})h,kh)$$

$$+ a_1((j+\tfrac{1}{2})h,kh).$$

When $j = 1$ or $k = 1$ we define $D_{j,k} = 0$ or $B_{j,k} = 0$ respectively. The following properties of A and the quantities defined in (2.3) will be used below.

$$(AX)_{j,k} = E_{j,k}X_{j,k} + B_{j,k}X_{j,k-1} + D_{j,k}X_{j-1,k} + D_{j+1,k}X_{j+1,k}$$

$$+ B_{j,k+1}X_{j,k+1}$$

$$B_{j,k} \leq 0 \quad , \quad B_{j,1} = 0$$

$$D_{j,k} \leq 0 \quad , \quad D_{1,k} = 0 \qquad (2.4)$$

and

$$E_{j,k} \geq -2(B_{j,k} + D_{j,k}) \text{ with equality holding if } B_{j,k} \neq 0$$

and $D_{j,k} \neq 0$ for $j = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, n$.

The subscript (j,k) refers to the gridpoint (jh,kh).

3. THE FACTORIZATIONS OF STONE

Stone has defined two factorizations both of which depend on a parameter α. One yields a symmetric auxiliary matrix and the other yields a nonsymmetric auxiliary matrix.

For the symmetric case, [4] the factors L and U of $A + B^{\alpha}$ are defined as

$$(LX)_{j,k} = b_{j,k}x_{j,k-1} + c_{j,k}x_{j-1,k} + d_{j,k}x_{j,k},$$
$$(UX)_{j,k} = x_{j,k} + e_{j,k}x_{j+1,k} + f_{j,k}x_{j,k+1}, \tag{3.1}$$

where

$$b_{j,k} = B_{j,k} - \alpha c_{j,k-1}f_{j-1,k-1},$$
$$c_{j,k} = D_{j,k} - \alpha b_{j-1,k}e_{j-1,k-1}, \tag{3.2}$$
$$d_{j,k} + b_{j,k}f_{j,k-1} + c_{j,k}e_{j-1,k} = E_{j,k} + \alpha c_{j,k-1}f_{j-1,k-1}$$
$$+ \alpha b_{j-1,k}e_{j-1,k-1},$$
$$d_{j,k}e_{j,k} = D_{j+1,k} - \alpha b_{j,k}e_{j,k-1},$$

and

$$d_{j,k}f_{j,k} = B_{j,k+1} - \alpha c_{j,k}f_{j-1,k}.$$

In addition we define

$$C_{j,k} = b_{j,k}e_{j,k-1} \text{ and } G_{j,k} = c_{j,k}f_{j-1,k}.$$

The nonsymmetric case [3] is more general in that it is not necessary to have A defined as in (2.4). To avoid any confusion the original matrix will be called M and the auxiliary matrix N. M may be nonsymmetric as is the matrix which naturally arises in the solution of a problem with Neumann boundary conditions. Let M be defined as follows:

$$(MX)_{j,k} = B_{j,k}X_{j,k-1} + D_{j,k}X_{j-1,k} + E_{j,k}X_{j,k}$$

$$+ F_{j,k}X_{j+1,k} + H_{j,k}X_{j,k+1}.$$

The auxiliary matrix N is defined by factors L and U which are in the form (3.1) just as in the symmetric factorization. Instead of (3.2) however, the elements of L and U are defined by

$$\begin{aligned} b_{j,k} &= B_{j,k} - \alpha C_{j,k}, \\ c_{j,k} &= D_{j,k} - \alpha G_{j,k}, \\ d_{j,k} + b_{j,k}f_{j,k-1} + c_{j,k}e_{j-1,k} &= E_{j,k} + \alpha C_{j,k} + \alpha G_{j,k}, \\ d_{j,k}e_{j,k} &= F_{j,k} - \alpha C_{j,k}, \\ d_{j,k}f_{j,k} &= H_{j,k} - \alpha G_{j,k}, \end{aligned} \tag{3.3}$$

where $C_{j,k} = b_{j,k}e_{j,k-1}$ and $G_{j,k} = c_{j,k}f_{j-1,k}$.

Since there will usually be no confusion as to whether the auxiliary matrix is defined by (3.2) or (3.3), B will be used to denote either of them.

With L and U of the form (3.1) A + B (or M + N) is given by

$$((A+B)X)_{j,k} = ((LU)X)_{j,k} = b_{j,k}x_{j,k-1} + b_{j,k}e_{j,k-1}x_{j+1,k-1}$$

$$+ c_{j,k}x_{j-1,k} + (d_{j,k} + b_{j,k}f_{j,k-1} + c_{j,k}e_{j-1,k})x_{j,k}$$

$$+ d_{j,k}e_{j,k}x_{j+1,k} + c_{j,k}f_{j-1,k}x_{j-1,k+1} + d_{j,k}f_{j,k}x_{j,k+1}.$$

4. WHY STONE'S FACTORIZATIONS ARE EXPECTED TO CONVERGE

In this section we explain why the factorizations defined in (3.2) and (3.3) with $\alpha = 1$ are expected to make the iteration

$$(A+B)X_{n+1} = (A+B)X_n - (AX_n - q)$$

rapidly convergent.

The matrix M which multiplies the error vector $E_n = X - X_n$ on each iteration is called the iteration matrix. It is well known that an iteration is convergent if the spectral radius of the iteration matrix is less than one. Moreover as the spectral radius decreases, the rate of convergence increases. Therefore we will show how the factorizations (3.2) and (3.3) are designed to produce an iteration matrix with a small spectral radius.

Throughout the following the eigenvalues of an $n \times n$ matrix M will be denoted by $\lambda_i(M)$ with

$$\lambda_{MAX}(M) = \lambda_1(M) \geq \lambda_2(M) \ldots \geq \lambda_n(M) = \lambda_{MIN}(M).$$

The iteration matrix of

$$(A+B)X_{n+1} = (A+B)X_n - (AX_n - q) \tag{4.1}$$

is $M = I - (A+B)^{-1}A$. The spectral radius of this matrix is small if the $\lambda_i((A+B)^{-1}A)$ are close to 1. Thus the iteration is rapidly convergent if $(A+B)^{-1}A$ is approximately equal to the identity I. That is if

$$\frac{\|X-(A+B)^{-1}AX\|}{\|X\|} \text{ is small for all } X \neq 0. \tag{4.2}$$

The factorization (3.3), suggested by Stone, attempts to satisfy (4.2) by making $\|(A+B)X - AX\|$ small in the sense that

$$(BX)_{j,k} = O(h^2) \tag{4.3}$$

where h is the meshsize of the grid used in the discretization of the differential problem, i.e., deriving (2.2) from (2.1).

To see that BX satisfies the condition (4.3) we have the following. Let $x = (x_{1,1}, \ldots, x_{n,n})^T$ where the $x_{j,k}$ are values of a smooth function taken at the gridpoints (jh,kh). If B is the matrix given by (3.3) and $C_{j,k}$ and $G_{j,k}$ are also as in (3.3), then comparing the definition of A in (2.4) to the definition of A+B in (3.3) and (3.4) yields

$$[(A+B)X]_{j,k} = (AX)_{j,k} + C_{j,k}\,[x_{j+1,k-1} + \alpha(x_{j,k}-x_{j+1,k}-x_{j,k-1}) \tag{4.4}$$

$$+ G_{j,k}\,[x_{j-1,k+1} + \alpha(x_{j,k}-x_{j-1,k}-x_{j,k+1})].$$

When $\alpha = 1$, Taylor's Theorem implies the bracketed terms of (4.4) satisfy

$$x_{j+1,k-1} + (x_{j,k}-x_{j+1,k}-x_{j,k-1}) = O(h^2)$$

and

$$x_{j-1,k+1} + (x_{j,k}-x_{j-1,k}-x_{j,k+1}) = O(h^2).$$

Stone's symmetric factorization (3.2) satisfies (4.2) in the sense that $(BX)_{j,k} = O(h)$. This also follows from applying Taylor's Theorem to $(BX)_{j,k} = ((A+B)X-AX)_{j,k}$.

5. WHY STONE'S FACTORIZATIONS DIVERGE WHEN NO PARAMETERS ARE USED

The meaning of small used above is weak since (4.2) and (4.3) are not equivalent. In fact, empirical results show that both factorizations (3.2) and (3.3) yield a B such that $\rho(I-(A+B)^{-1}A) > 1$ when A is derived from the Laplacian [$a_1(x) \equiv a_2(x) \equiv 1$ in (1.1)] and $n = 30$. Thus even though $(BX)_{j,k} = O(h)$ or $O(h^2)$, $\|X-(A+B)^{-1}AX\|$ is not small. It isn't even less than 1. In this section we will see how the above situation is possible.

We shall begin by deriving some properties of the eigenvalues of $(A+B)^{-1}A$.

Lemma 5.1:

If A is given by (2.4) and B is defined by either of Stone's factorizations, (3.2) or (3.3), then one is an eigenvalues of $(A+B)^{-1}A$.

Proof:

If B is given by (3.2) or (3.3) then the first row and column are null. Hence B has a zero eigenvalue. Let w be the corresponding eigenvector. Then $(A+B)w = Aw$ and consequently $(A+B)^{-1}Aw = w$ which completes the proof.

Theorem 5.1:

The vector w is an eigenvector of $(A+B)^{-1}A$ iff $Bw = 0$ or $Aw = \sigma Bw$ where σ is a scalar other than 0 or -1. Furthermore the corresponding eigenvalue is $\frac{\sigma}{\sigma+1}$ if $Bw \neq 0$ and is 1 if $Bw = 0$. (5.1)

Proof:

Suppose $Aw = \sigma Bw$ $\sigma \neq -1, 0$. Then

$$w + A^{-1}Bw = w + \frac{1}{\sigma}w = (\frac{\sigma+1}{\sigma})w.$$

Hence

$$(A+B)^{-1}Aw = (A+B)^{-1}A\left(\frac{\sigma}{\sigma+1}\right)(w+A^{-1}Bw)$$

$$= \left(\frac{\sigma}{\sigma+1}\right)(A+B)^{-1}(Aw+Bw) = \frac{\sigma}{\sigma+1}w.$$

Now suppose $(A+B)^{-1}Aw = \frac{\sigma}{\sigma+1}\, w$, $\sigma \neq -1, 0$. Then

$$Aw = \frac{\sigma}{\sigma+1}(A+B)w$$

and therefore $Aw = \sigma Bw$.

The case $Bw = 0$ was taken care of in Lemma 5.1 and the proof is therefore complete.

For simplicity we shall now consider only the symmetric factorization (3.2). The eigenvalues $\lambda_i = \lambda_i(I-(A+B)^{-1}A)$ are real and have magnitude greater than one iff

$$|\lambda_i| = \left|1 - \frac{\sigma_i}{\sigma_i + 1}\right| \text{ where } -2 \leq \sigma_i \leq -\frac{1}{2}.$$

If A is defined by (2.4) and (A+B) is defined by (3.2) then it is easily shown that $\lambda_i((A+B)^{-1}A) > 0$, see Diamond [1]. This and Theorem 5.1 imply $\frac{\sigma_i}{\sigma_i + 1} > 0$. Thus if $\sigma_i < -\frac{1}{2}$ then $\sigma_i + 1$ must also be negative and $\sigma_i < -1$. Hence the iteration (4.1) is divergent if there exists some w such that $Aw = \sigma Bw$, $-2 \leq \sigma \leq -1$. Moreover as σ approaches -1, $\left(1 - \frac{\sigma}{\sigma + 1}\right)$ approaches infinity and the iteration diverges rapidly, i.e. the error grows rapidly.

The iteration is rapidly convergent when $|\lambda_i|$ is small or, in other words, when $|\sigma_i|$ is large. The fact that $(Bw)_{j,k}$ is small is therefore only significant if it is small relative to $(Aw)_{j,k}$. If for some w $\|Aw\|$ is small, say $O(h^2)$, then selecting a B which satisfies (4.3) does not guarantee (4.2). This difficulty arises on even the simplest problems as seen from the following considerations. It is well known (see for example Frankel [2] that for A the $(n^2 \times n^2)$ matrix derived from the Laplacian,

$$\lambda_{min}(A) \cong \frac{.2}{2(n+1)^2} = O(h^2) \text{ where } h = \frac{1}{n+1}.$$

Thus in this case the fact that B as defined by Stone satisfies $(BX)_{j,k} = O(h)$ or $O(h^2)$ doesn't imply that $\|X - (A+B)^{-1}AX\|$ is small for all $X \neq 0$.

6. EMPIRICAL RESULTS CONCERNING λ_{MAX}

In this section we shall present some empirical results which indicate $\lambda_{MAX}((A+B)^{-1}A)$ grows without bound as the dimension of A increases. This is done by showing the existence of vectors w^n, where the n corresponds to the dimension of A, such that

$$\lim_{n\to\infty} \frac{< Aw^n, w^n >}{< (A+B)w^n, w^n >} = \infty .$$

Then using the equality

$$\lambda_{MAX}((A+B)^{-1}A) = \max_{\lambda \neq 0} \frac{< AX, X >}{< (A+B)X, X >} \tag{6.1}$$

we conclude $\lim_{n\to\infty} \lambda_{MAX}((A+B)^{-1}A) = 0.$

The results of Section 5 suggest the vectors w^n will satisfy

$$Aw^n = \sigma_n Bw^n \tag{6.2}$$

where

$$\lim_{n\to\infty} \sigma_n = -1.$$

The vectors do not satisfy (6.2). However, as n increases

$$(Aw^n)_{j,k} \cong -(Bw^n)_{j,k} \text{ for } j \neq 1,n \text{ and } k \neq 1,n;$$

and

$$< Aw^n,\ w^n > = \sigma_n < Bw^n, w^n >$$

where (6.3)

$$\lim_{n\to\infty} \sigma_n = -1.$$

From (6.3) we see that

$$\frac{< Aw^n, w^n >}{< (A+B)w^n, w^n >} = \frac{1}{1 + \frac{1}{\sigma_n}} = \frac{\sigma_n}{1 + \sigma_n}$$

and this approaches infinity as $\sigma_n \to -1$. Furthermore the vectors w^n do not maximize

$$\frac{< AX, X >}{< (A+B)X, X >} .$$

Instead they give a lower bound for

$$\max_{X\neq 0} \frac{< AX, X >}{< (A+B)X, X >}$$

and thus, from (6.1), they give a lower bound for $\lambda_{MAX}((A+B)^{-1}A)$.

The matrix A is derived from the Laplacian and B is the auxiliary matrix defined in (3.2) with $\alpha = 1$. (The results also apply to the auxiliary matrix defined in (3.3) since the elements of the two matrices approach the same values as the order of A increases. This can be seen by noting that the proof of Theorem 5.1 in [1] can be applied to the elements of the nonsymmetric factorization as well as the symmetric factorization.)

A number of different vectors were tried on a trial and error basis in attempting to find a w^n such that $< Aw^n,\ w^n > \cong - < Bw^n,\ w^n >$. The process of selecting vectors was started by using the algorithm of Diamond [1] to

calculate the eigenvector w corresponding to the largest eigenvalue of $(A+B)^{-1}A$. Next a function $f(x_1,x_2)$ was found such that $f(jh,kh) \cong w_{j,k}$. A vector $F_1 = (f_{1,1}{}^1, \ldots, f_{n,n}{}^1)$, where $f_{j,k}{}^1 = f(jh,kh)$ and $h = 1/(1+n)$, was defined and used as the first trial vector. A more or less random procedure was employed to alter the trial vector F_i and form the succeeding vector F_{i+1}. The final vector is denoted by w^n, where as before the n corresponds to the dimension of A.

From the eigenvector, which was calculated to select the first trial vector, the following information was inferred about the function $f(x_1,x_2)$.

1. The function should have a maximum at the center of the region, $(\frac{1}{2}, \frac{1}{2})$.
2. The function is symmetric with respect to the lines $x_2 = -x_1$, $x_2 = \frac{1}{2}$, $x_2 = 1 - x_1$ and $x_1 = \frac{1}{2}$.
3. The function approaches zero as (x_1,x_2) approaches the boundary.
4. The function decreases faster along the line $x_2 = x_1$ than along $x_2 = 1 - x_1$.

Functions with the above properties were tried with the additional possibility that they were periodic in the x_1 and x_2 directions with period π/m , m an integer. In this case the function exhibited the above properties in each period.

The intention was to determine $< Aw^n, w^n >$ and $< Bw^n, w^n >$ when n was very large. That is when B may be partitioned as

$$B = \begin{pmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{pmatrix}$$

where $B_{2,2}$ is an $(N^2 \times N^2)$ submatrix with almost constant diagonals. Using the limiting values of the elements of B as derived in [1] we see that

$$B_{2,2} \cong B_{2,2}^{*} = \begin{pmatrix} 2 & -1 & & 1 & -1 & & \\ -1 & 2 & -1 & & 1 & -1 & \\ & -1 & 2 & -1 & & 1 & -1 \\ -1 & & -1 & 2 & -1 & & 1 \\ & -1 & & -1 & 2 & -1 & \\ & & -1 & & -1 & 2 & -1 \\ & & & -1 & & -1 & 2 \end{pmatrix}$$

Instead of computing $< Bw^n, w^n >$ the quantity $< B_2^{*}, {}_2w^N, w^N >$ was computed. We have $< B_2^{*}, {}_2w^N, w^N > \cong < B_2, {}_2w^N, w^N > = < Bw^n, w^n >$ where

$$w^n = \begin{matrix} 0 \\ w^N \end{matrix}$$

and 0 is an (n^2-N^2)-component zero vector. Similarly $< Aw^n, w^n >$ was computed from $< Aw^n, w^n > = < A_{2,2}w^N, w^N >$ where

$$A = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix}.$$

Then instead of computing

$$\sigma_n = \frac{< Aw^n, w^n >}{< Bw^n, w^n >} = \frac{< A_{2,2}w^N, w^N >}{< B_{2,2}w^N, w^N >},$$

the quantity

$$\sigma_N^* = \frac{< A_{2,2} w^N,\ w^N >}{< B_{2,2}^* w^N,\ w^N >}$$

was computed.

The vector w^N, as described above, was chosen to be

$$w_{j,k}^N = \exp\{-(|jh-\tfrac{1}{2}| + |kh-\tfrac{1}{2}| + 1)^2\ [2 - \cos((jh-kh)\tfrac{3\pi}{2})]\}$$

$$\cos^{20}((jh-kh)\tfrac{3\pi}{2}).$$

For N = 100 and N = 250 the values of σ_N^* were $\sigma_{100}^* = -1.0918$ and $\sigma_{250}^* = -1.0661$. Values of N between 100 and 250 produced values of σ_N^* which varied monotonicly from σ_{100}^* to σ_{250}^*. This indicates that as N increases σ_N^* gets close to -1 and

$$\frac{1}{1 + \sigma_n} \cong \frac{1}{1 + \sigma_N^*}$$

grows without bound. Thus $\lambda_{MAX}((A+B)^{-1}A)$ grows without bound as the dimension of A increases.

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